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SHORTER COMMUNICATIONS

THERMAL INSTABILITY OF A HORIZONTAL LAYER OF NON-NEWTONIAN FLUID HEATED FROM BELOW

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NOMENCLATURE

A,	area;
a,	dimensionless wave number relating to critical Rayleigh number;
$A_1^{(m)}, A_2^{(m)}, B_1^{(m)}, B_2^{(m)}$,	coefficients of equation (16);
C_m,	coefficient of equation (14);
d,	height of liquid layer [cm];
D,	differential operator, defined as d/dx_3^+;
f,	function associated with W and H, equations (10a, 10b);
g,	gravitational acceleration [cm/s^2];
h,	heat-transfer coefficient;
H,	quantity associated with temperature disturbance, equation (10a);
H^*,	quantity associated with H, equation (18);
k,	thermal conductivity;
m,	consistency index in Power-law model [g s^{n}cm^{-2}];
n,	flow behavior in Power-law model;
N_{Ra},	Rayleigh number, defined by equation (9);
$(N_{Ra})_{cr}$,	critical Rayleigh number;
Q,	amount of thermal energy transferred across liquid layer;
Q_{total},	total power input as registered by Wattmeter;
T,	temperature [°C];
ΔT,	temperature difference between lower and upper plates;
u_m,	quantity related to W, according to equation (15);
$\mathbf{v}_i$,	velocity vector [cm/s];
$\mathbf{v}^+$,	dimensionless velocity vector as defined in equation (8b);
$\mathbf{v}^*$,	dimensionless velocity vector as defined in equation (19);
w,	velocity component along x_3 direction [cm/s];
w^+,	dimensionless velocity component along x_3 direction;
W,	quantity associated with velocity disturbance, see equation (10b);
x_i,	coordinate [cm].

Greek symbols

α,	thermal expansion coefficient [1/°C];
Δ_{ij},	rate of deformation tensor [1/s], see equation (4);
Δ_{ij}^*,	dimensionless rate of deformation tensor, see equation (8d)
ε_v,	kinetic energy dissipated by the viscous force;
ε_g,	internal energy released due to buoyancy force;
κ,	thermal diffusivity [cm^2/s];
ρ,	density [g/cm^3];
τ,	stress tensor;
τ^+,	dimensionless stress tensor, see equation (8e);
τ^*,	dimensionless stress tensor, see equation (21);
θ,	temperature disturbance [°C];
θ^+,	dimensionless temperature disturbance.

THE STABILITY criterion based on linear analysis for the onset of convection of a horizontal liquid layer with linear temperature–density relationship was first given by Rayleigh

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[5]. Rayleigh showed that the onset of convection occurs if the value of a certain dimensionless parameter (i.e. Rayleigh number) exceeds its critical value. Rayleigh number is defined as

$$N_{Ra} = \frac{g\alpha\,\Delta T\,.\,d^3}{\kappa\nu} \tag{1}$$

Rayleigh's analysis was made for the case of two free boundary surfaces. Subsequent investigators [3, 4] have considered more realistic situations. An excellent account of this problem can be found in the treatise of Chandrasekhar [1].

The heightened interests in non-Newtonian fluid displayed of recent years make it natural to extend this stability study to non-Newtonian systems. Besides its academic interest, study of this type will also be important in terms of its rheological significance. The observation of the onset of convection provides a potentially useful way of determining the limiting viscosity. The stability analysis carried out in this work will also be useful in determining the adequacy of a given rheological model in describing the free convection phenomenon.

In carrying out the present work, it was decided to use Powerlaw model for the characterization of non-Newtonian behavior. Although many objections have been raised against its use, the inescapable fact remains that this model gives a reliable representation for an important aspect of non-Newtonian behavior (i.e. variable viscosity) for a large number of systems over a wide range of shear rates. Furthermore, its relative simple form facilitates the necessary computation work.

The classical approach in carrying out the stability analysis is the use of linear stability theory. This approach, however, is not suitable to non-Newtonian fluid with nonlinear constitutive equations. Another possible approach is the use of finite differences method. However, for a non-Newtonian fluid, the difference equations of the equation of motion would be highly nonlinear, and requires excessive computation. This, in turn, may make the method impractical.

A somewhat less involved method can be developed based upon the thermodynamic significance of the critical Rayleigh number obtained from the linear stability theory for Newtonian fluids. Chandrasekhar [1] stated:

"Instability occurs at the minimum temperature gradient at which a balance can be steadily maintained between the kinetic energy dissipated by viscosity and the internal energy released by the buoyancy force."

Although this statement is based upon the result of linear stability theory and for Newtonian fluid, it is plausible to assume that this is at least approximately correct for all fluids. Based on this hypothesis a stability criterion for non-Newtonian fluid can be developed. This is discussed in later sections.

ANALYSIS

The analysis given in this work is concerned with a horizontal layer of incompressible non-Newtonian fluid of depth d and confined between two horizontal parallel surfaces imposed at a temperature of T_1 (at lower surface) and T_2 (at upper surface), respectively. The equation of state of the fluid and the rheological equation of state are assumed to be respectively as

$$\rho = \rho_0[1 - \alpha(T - T_0)] \tag{2}$$

$$\tau = m[\sqrt{(\tfrac{1}{2}\Delta:\Delta)}^{n-1}]\,\Delta. \tag{3}$$

The meaning of symbols are given in nomenclature. In Cartesian tensor notation, Δ is given as

$$\Delta_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}. \tag{4}$$

For a column of fluid with unit cross section, the kinetic energy dissipated by the viscous force, ε_v, and the internal energy released due to buoyancy force, ε_g, can be expressed as (2):

$$\varepsilon_v = -\int_0^d \langle \mathbf{v}\,.\,(\nabla\,.\,\tau)\rangle\,\mathrm{d}x_3 \tag{5}$$

$$\varepsilon_g = \rho g\alpha\int_0^d \langle \theta w\rangle\,\mathrm{d}z = -\frac{\rho g\alpha\kappa g}{\Delta T}\int_0^d \langle \theta\nabla^2\theta\rangle\,\mathrm{d}x_3 \tag{6}$$

$$\Delta T = T_1 - T_2 \tag{7}$$

The x_3 coordinate is measured as distance away from the lower surface (i.e. $z = 0$, at lower surface). The brackets, $\langle\ \rangle$, refers to the average quantity over a given x–y plane.

First, the following dimensionless quantities are introduced:

$$\theta^+ = \frac{\theta}{\Delta T} \tag{8a}$$

$$\mathbf{v}^+ = \frac{\mathbf{v}d}{\kappa} \tag{8b}$$

$$x_i^+ = \frac{x_i}{d} \tag{8c}$$

$$\Delta_{ij}^+ = \frac{\partial \mathbf{v}_i^+}{\partial x_j^+} + \frac{\partial \mathbf{v}_j^+}{\partial x_i^+} \tag{8d}$$

$$\tau^+ = \{\sqrt{[\tfrac{1}{2}(\Delta^+ : \Delta^+)]}\}^{n-1}\,\Delta^+. \tag{8e}$$

By equating equations (5) and (6) and expressing the results in terms of the dimensionless quantities, after some rearrangement, one has

$$N_{Ra} = \frac{\rho g\alpha(\Delta T)\,d^{2n+1}}{\kappa_m^n} = \frac{\int_0^1 \langle \mathbf{v}^+\,.\,[\Delta^+\,.\,\tau^+]\rangle\,\mathrm{d}x_3^+}{\int_0^1 \langle \theta^+\nabla^{+2}\theta^+\rangle\,\mathrm{d}x_3^+}. \tag{9}$$

equation (9), provided suitable expressions for v^+ and θ^+ number. If the thermodynamic principle on the onset of convection, stated in the previous section, is assumed to be valid, the critical Rayleigh number can be evaluated from Equation (9), provided suitable expressions for v^+ and θ^+ at marginal state are available.

The marginal expression can be written as (2):

$$\theta^+ = H(x_3^+)\, f(x_1^+, x_2^+) \tag{10a}$$

$$w^+ = v_3^+ = W(x_3^+)\, f(x_1^+, x_2^+) \tag{10b}$$

where f is a two-dimensional periodic function of x_1^+ and x_2^+. The other velocity components, according to [2], are:

$$v_1^+ = \frac{v_1}{(\kappa/d)} = \frac{DW}{a^2} f_{x^+_1}. \tag{10c}$$

$$v_2^+ = \frac{v_2}{(\kappa/d)} = \frac{DW}{a^2} f_{x_2^+} \tag{10d}$$

The operator, D, denotes d/dx_3^+. The subscript x_1^+ refers to the partial differentiation with respect to the variable.

If one assumes that the marginal solutions of the Newtonian case is applicable, H and W are solutions of the following differential equations:

$$N_{Ra} = \frac{\rho g \alpha (\Delta T) d^{1+2n}}{\kappa^n m} = \frac{-C_1^{n-1}\left[\{(N_{Ra})_{cr}\, a^2\}_{\text{Newtonian}}\right]^{n+1} \int_0^1 \langle v^* . (\nabla^+ . \tau^*)\rangle \, dx_3^+}{\langle f^2 \rangle \int_0^1 \left[(DH^*)^2 + a^2 H^{*2}\right] dx_3^+} \tag{20}$$

$$(D^2 - a^2)^3 H = -(N_{Ra})_{cr} a^2 H \tag{11}$$

$$(D^2 - a^2) H = -W \tag{12}$$

with the boundary conditions:

$$H = W = 0, \quad x_3^+ = 0 \text{ and } 1 \tag{13a}$$

$$DW = 0, \quad x_3^+ = 0 \text{ and } 1. \tag{13b}$$

The critical Rayleigh number is given as 1707·76 at $a = 3{\cdot}117$. H and W are given as:

$$H = \sum_{m=1}^{\infty} C_m \sin m\pi x_3^+ \qquad W = (N_{Ra})_{cr} a^2 \sum_{m=1}^{\infty} C_m u_m \tag{14}$$

$$= (1707{\cdot}76)(3{\cdot}117)^2 \sum_{m=1}^{\infty} \frac{C_m}{(m^2\pi^2 + a^2)^2} \left[A_1^{(m)} \cosh a x_3^+ + A_2^{(m)} x_3^+ \cosh a x_3^+ + B_1^{(m)} \sinh a x_3^+ + B_2^{(m)} x_3^+ \sinh a x_3^+ + \sin m\pi x_3^+\right] \tag{15}$$

The coefficients, $A_1^{(m)}$, etc., are given as

$$A_1^{(m)} = 0 \tag{16a}$$

$$B_1^{(m)} = \frac{m\pi}{\Delta}\left[a - (-1)^{m+1} \sinh a\right] \tag{16b}$$

$$A_2^{(m)} = -\frac{m\pi}{\Delta}\left[\sinh^2 a - (-1)^{m+1} a \sinh a\right] \tag{16c}$$

$$B_2^{(m)} = \frac{m\pi}{\Delta}\left[(\sinh a \cosh a - a) - (-1)^{m+1} (a \cosh a - \sinh a)\right] \tag{16d}$$

and

$$\Delta = \sinh^2 a - a^2. \tag{17}$$

Two additional dimensionless quantities will be introduced here. They are:

$$H^* = \frac{H}{C_1} = \sum_{m=1}^{\infty} \left(\frac{C_m}{C_1}\right) \sin m\pi x_3^+ \tag{18}$$

$$v^* = \frac{v^+}{C_1 (N_{Ra})_{cr} a^2}. \tag{19}$$

Combining equations (9), (14), (15), (18) and (19), one has,

and

$$\tau^* = \left|\sqrt{(\tfrac{1}{2}\Delta^* : \Delta^*)}\right|^{n-1} \Delta^* \tag{21}$$

$$\Delta_{ij}^* = \frac{\partial v_i^*}{\partial x_j^+} + \frac{\partial v_j^*}{\partial x_i^+}. \tag{22}$$

Equation (20) provides the basis of calculation of the critical Rayleigh number. The detailed description is given in the next section.

COMPUTATION WORK

Before proceeding with the computation of critical Rayleigh number, the cell pattern of flow prevailed at the onset of convection needs to be known. This means the functional form of f has to be specified. Two cases will be considered. They are:

(a) *Two-dimensional roll*—f is given as:

$$f = \cos a\, x_1^+ \tag{23}$$

and

$$\langle f^2 \rangle = \frac{\int_0^{2\pi/a} \cos^2 a\, x_1^+\, dx_1^+}{\left(\frac{2\pi}{a}\right)} = \frac{1}{2} \tag{24}$$

$$f_{x_1^+} = -a \sin x_1^+, \quad f_{x_2^+} = 0. \tag{25}$$

(b) *Three-dimensional hexagonal cell*—For this case, f is given as Christopherson [2],

$$f^+(x_1^+, x_2^+) = \frac{1}{2}\left[2\cos\left(\frac{\sqrt{3}}{2} a\, x_1^+\right)\cos\left(\frac{a}{2} x_2^+\right) + \cos a\, x_2^+\right]. \tag{26}$$

The length of the side of the hexagonal cell, L, is related to the wave number, a, as

$$a = \frac{4\pi}{3L} \tag{27}$$

and

$$\langle f^+ \rangle = \frac{\int_0^{(\frac{3}{2})L} \int_0^{x_1^+/\sqrt{3}} \left(\frac{1}{9}\right)\left[2\cos\frac{2\pi x_1^+}{\sqrt{3}}\cos\frac{2\pi}{3L}x_2^+ + \cos\frac{4\pi}{3L}x_2^+\right]^2 dx_1^+\, dx_2^+}{\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}L\right)\left(\frac{L}{2}\right)} = \frac{1}{6}. \tag{28}$$

Once f^+ is known, the dimensionless velocity components v_i^* can be obtained from equations (10a)–(10c) and equation (19), from which the rate of deformation tensor can be evaluated accordingly. This, in turn, enables the calculation of τ^* in terms of f^+ and W [equations (21) and (22)]. Once these quantities are known, they can be substituted into equation (20) for the evaluation of critical Rayleigh number. The integration is carried out using Simpson's rule and using $x_3^+ = \frac{1}{60}$. The infinite series is approximated by taking $m = 4$. For both cases, the computation was made for n ranging from unity to 0·4.

It should be mentioned that the coefficient C_m's for the expression of the amplitude of temperature deviation, H, [see equation (14)] can be determined only to a factor, i.e. the results of linear theory yield expressions of C_m/C_1. It is arbitrarily decided to take $C_1 = 1$. The justification of this assumption can only be made in terms of agreement between the predicted value of the critical Rayleigh number and experimental results. This will be discussed in later sections.

Numerical values of the critical Rayleigh number vs the flow behavior index obtained for both cases are shown in Fig. 1. For the special case of $n = 1$, the predicted value of the critical Rayleigh number are 1707·57 for both cases. The predicted value according to linear stability theory is 1707·76. This close agreement seems to indicate that the numerical procedure involved in this calculation is of sufficient accuracy.

For values of n other than unity, the critical Rayleigh number computed on the basis of two-dimensional roll is found to be consistently lower than that based on hexagonal cell. The critical Rayleigh number is known to be independent of flow pattern. This discrepancy may be caused either by the inadequacy of the basic assumption of the thermodynamic significance on the onset of convection or due to the use of marginal state solutions for Newtonian case or both. The difference between these results is only moderate and, on the average, is of the order of 10 per cent which is comparable with the experimental work of Schmidt and Milverton (6) for Newtonian fluid, but somewhat inferior to those of Silverton (8). This rather moderate difference seems to indicate that the results obtained in this work is of sufficient accuracy and can be used for predicting purposes.

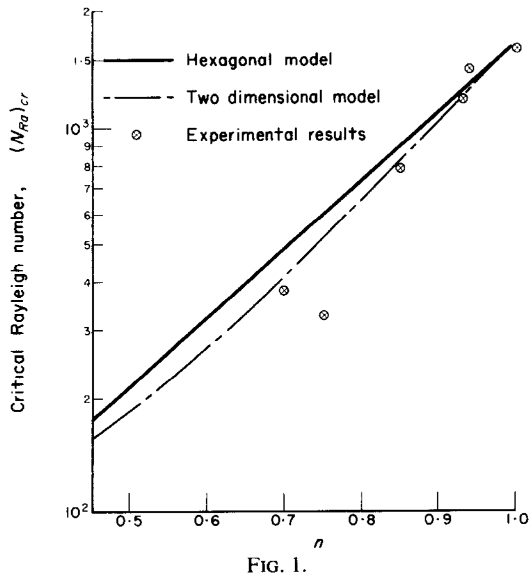

FIG. 1.

EXPERIMENTAL WORK

Experimental study was carried out to verify the results of the approximate stability analysis given above. The detection of the onset of convection of a horizontal layer of Newtonian fluid heated from below has been studied experimentally by a number of investigators [6–8]. Most of these studies were based on the principle developed by Schmidt and Milverton [6] which can be briefly described as follows: consider a horizontal liquid layer with depth d and heated from below. If the temperature difference between the lower and upper surfaces, ΔT, is plotted against the amount of thermal energy across the layer, Q, a linear relationship would be obtained as long as the mode of heat transfer is conduction. The slope of this straight line is simply given as $\Delta T/Q = d/Ak$, where k is the thermal conductivity of the fluid A in the surface area of the layer. On the other hand, if convective motion prevails within the liquid layer, the quantity $\Delta T/Q$ becomes $\Delta T/Q = 1/Ah$, where h is the heat-transfer coefficient. Since $1/h$ is always less than that of d/k, an abrupt change in the slope of the curve ΔT vs. Q indicates the commencement of convective motion.

Apparatus

The experimental apparatus was designed to approximate the idealized situation of a horizontal layer of liquid of infinite extent confined between two rigid horizontal surfaces with heating from below. The test chamber consists of a rectangular space. The base of the chamber is provided by an electrical heater plate and the sides by four walls which are formed by cutting a rectangular hole 8 by 6 in. in a $\frac{1}{2}$ in. thick piece of phonolitic plastic. The upper surface is provided by a $\frac{1}{4}$ in. thick glass plate. The glass plate is carried on ball bearings resting in slots which are placed into positions cut along the four sides of the rectangular holes of the phonolitic plate. By using slots of different heights, the depth of the liquid layer can be changed. The lower surface temperature is measured by copper-constantan thermocouples embedded in the heater plate at a depth of $\frac{1}{64}$ in. below the metal–liquid interface. The temperature of the glass–liquid interface was determined by two thermocouples which are placed through small holes drilled through the glass plate. The detailed description of the apparatus is given elsewhere [9] and will not be repeated here. The schematic diagram of the apparatus is shown in Fig. 2.

Material

Aqueous solutions of carboxy-methyl-cellulose were used in this work which included 0·75 % CMC-74, 1·2 % CMC-74, 5 % CMC-7L, 5·5 % CMC-7, and 7 % CMC-7L2 manufactured by Hercules Chemical Company. In addition, glycerin was used in the preliminary measurement in order to check the accuracy of the apparatus.

For the evaluation of critical Rayleigh number from experimental data, a number of physical properties need to be determined. The rheological properties of the aqueous CMC solutions were determined using a Brookfield Synchro-Lectric viscometer over a shear range of 10^{-1} to 10^{1} sec^{-1} and temperature range of 20–60°C, and they are fitted with Power-law model. The thermal conductivity was determined with an apparatus of Briggs' type. The thermal expansion coefficient was obtained from density–temperature relationship with the density data obtained through the use of a pycnometer. Heat capacities were also measured but were found to differ only slightly from those of water. The details of these determinations are described in [9].

RESULTS

For a given series of experiments, eight to ten sets of readings were obtained. These readings were equally divided into conduction and convection. The temperature difference from a few degrees (°C) up to 16°C was maintained, corresponding to a power input up to 40 W. The average time required for each reading was from a few hours up to 20 hr. It was found that the attainment of steady-state was greatly influenced even by very small change in ambient temperature.

For each series of measurements, a set of readings of Q_{total} vs. ΔT was obtained. The quantity, Q_{total}, represents the

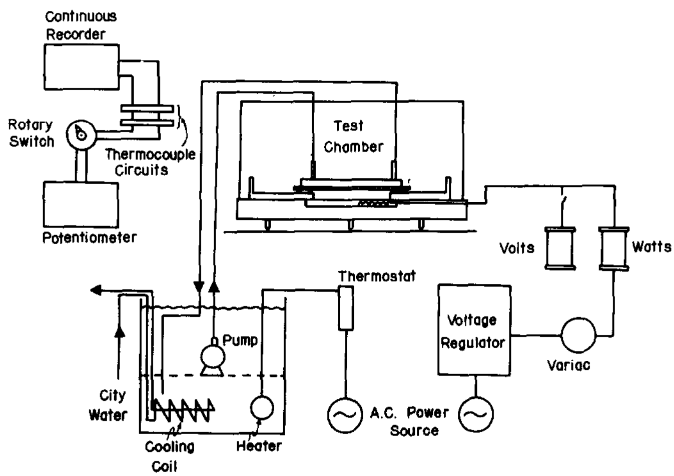

FIG. 2.

total power input registered by the wattmeter and is the sum of the thermal energy transmitted across the liquid layer and that dissipated to the surroundings through the underside and edges of the heater plate. However, as shown in earlier investigations [6, 7], for the purpose of detecting the change in slope of the curve of Q vs. ΔT, it is not necessary to correct for the heat loss and one can use the quantity, Q_{total}, directly. A typical set of data is shown in Fig. 3.

A summary of all experimental results is given in Table 1. The temperature difference, ΔT, corresponding to the onset of thermal instability is obtained by drawing best-fit lines visually through the data points and the intersection points of the two linear segments give the values of ΔT at onset of convection. The physical properties for calculation $(N_{Ra})_c$, equation (9), were evaluated at the average temperature of the liquid layer [i.e. $T_{av.} = (T_1 + T_2/2)$]. Comparison with theoretical results is shown in Fig. 3. The experimental results agree substantially with the results based upon the approximate analysis with the exception of one case (5·5% CMC-7L). The error involved is the experimental observation and can be judged by the comparison of the values of critical Rayleigh number obtained for the case of glycerin vs. the accepted theoretical value of 1706. It, therefore, appears to be plausible that the (at least a large part of) difference between the experimental results and theoretical values can be attributed to experimental errors.

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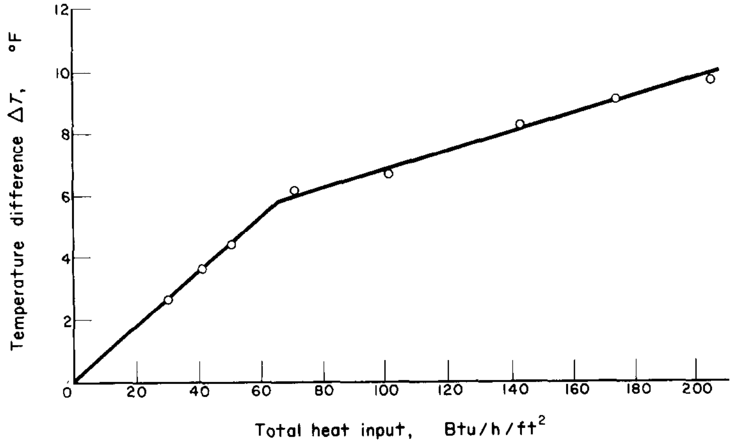

Fig. 3.

Table 1. Summary of experimental data

Solution	Glycerin	0·75%-CMC-7H	1·2%-CMC-7H	5%-CMC-7L	7%-CMC-7L2	5·5%-CMC-7L
d (ft)	0·0521	0·073	0·0875	0·0875	0·0521	0·0965
T_m (°F)	91·5	89·5	109·0	114·0	81·0	86·4
ΔT (°F)	5·05	3·51	10·4	10·61	15·8	13·8
n	1·0	0·94	0·85	0·7	0·93	0·75
$m \times 10^{-3}$/lbm/hft	1·19	1·331	9·34	2·14	1·333	2·866
k, Btu/hft°F	0·165	0·353	0·375	0·391	0·367	0·334
C_p, Btu/lbm°F	0·589	1·0	1·0	1·0	1·0	1·0
ρ, lbm/ft^3	78·45	62·17	63·3	63·08	64·07	63·60
$\beta \times 10^4$/1/°R	3·0	3·06	2·46	2·74	1·5	1·77
Ra_c	1621·6	1420·0	776·7	372·0	1220·0	329·45